# Multiplicity in Cascade Transmission Line Synthesis—Part II

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*Abstract*—The first portion of this paper[1] related to an examination of the synthesis of cascade transmission line structures providing a polynomial insertion loss function phrased in terms of the cosine of an electrical length. Certain features of nonuniqueness of the synthesis were uncovered which, particularly in the case of couplers, lead to a flexibility in design. This portion of the paper includes a specific examination of couplers and concludes with some general comments on transmission line synthesis.

## VI. Transmission Line Couplers

A TRANSMISSION line coupler is composed of two multisection conductors having reflection symmetry to one another about a longitudinal axis. The two conductor system possesses symmetric and antisymmetric modes with the symmetric characteristic impedance for each section always being the higher one for the two modes. If the structure is designed so that the symmetric and antisymmetric networks are mutually dual, namely that every characteristic impedance of the symmetric network corresponds to its inverse in the antisymmetric network, then a hybrid structure results. The hybrid action results from the opposed, but equal magnitude, reflections from the two networks and from their equal transfer functions. Signal fed into one port admixes into the two modes and the reflections interfere destructively back to the signal port and constructively to an adjacent port. There is no relative change in the modal admixture of the transmitted wave since both modal transfer functions are identical, so that transmitted output occurs on the same conductor as that of the input signal. There is no output at the fourth port because of symmetry-antisymmetry mode interference.

It suffices to consider just one of the two modes. We shall consider only the symmetry mode, where its reflection factor is equal to the scattering coefficient to the coupled port and the transfer function is the scattering coefficient to the transmission port. This result is shown in Fig. 3 (see Part I) for a unit input wave amplitude.

Often, the criterion of merit of a coupler is the flatness of coupling as a function of frequency which finds alternative phrasing in terms of the flatness of the joint loss function of the symmetric and antisymmetric networks. It is not infrequent, however, that phase requires consideration, as in quadrature hybrids, and this too enters into the synthesis formulation.

Let us consider now only the specification on coupling magnitude. We desire a reactive network possessing substantially constant reflection over a large frequency range. The network element we seek, ideally, is the ideal transformer which we may realize with good approximation to within an additive transmission line length over a broadband as a multisection quarter-wave transformer.

The imperfect approximation of the finite quarter-wave transformer to an ideal transformer may be represented in the fashion shown in Fig. 4. Here the quarter-wave transformer is represented as a line length plus the transformer $1/N$ plus still an added fourpole representing the fluctuation of the representation. The fourpole has an identity representation plus small perturbation terms and is characterized as follows

$$T_\epsilon = \begin{pmatrix} 1+\alpha & i\beta \\ i\gamma & 1+\delta \end{pmatrix} \tag{35}$$

where $\alpha$, $\beta$, $\gamma$, and $\delta$ are all small quantities. If the quarter-wave transformer is terminated by an $N$:1 transformer, the network is essentially matched and, neglecting the additive line length, we obtain

$$k_N \simeq \frac{1}{2}\left[(\alpha-\delta) + i\left(\frac{\beta}{N^2} - \gamma N^2\right)\right] \tag{36}$$

so that

$$|k_N|^2 \simeq \frac{1}{4}\left[(\alpha-\delta)^2 + \left(\frac{\beta}{N^2} - \gamma N^2\right)^2\right]. \tag{37}$$

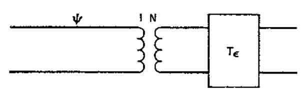


Fig. 4. Quarter-wave transformer equivalent.

If the quarter-wave transformer is terminated by a unit load the reflection is essentially equal to

$$k_1 \simeq -\left(\frac{N^2-1}{N^2+1}\right) \cdot\left(1 - \frac{2N^2}{N^4-1}\left[(\alpha-\delta) + i(\beta-\gamma)\right]\right) \tag{38}$$

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so that

$$|k_1|^2 \simeq \left(\frac{N^2-1}{N^2+1}\right)^2\left[1-\frac{4N^2}{N^4-1}(\alpha-\delta)+4\left(\frac{N^2}{N^4-1}\right)^2(\beta-\gamma)^2\right]. \quad (39)$$

Examination of (37) and (39) shows that the quarter-wave transformer operating into the appropriate mismatch to provide an overall network match produces power reflection fluctuations about the matched condition which is of second degree in all the variables $\alpha$, $\beta$, $\gamma$, and $\delta$. The quarter-wave transformer operating as a constant mismatch, however, produces first-order fluctuation terms in $\alpha$ and $\delta$, which severely limits usefulness.

While a quarter-wave transformer tends to be the design we seek for coupler instrumentation, nevertheless, there must be a significant modification to minimize the value of $(\alpha-\delta)$ compared to $(\beta-\gamma)$. Even if we do not obtain the quarter-wave transformer precisely in design we, nonetheless, expect to observe the characteristic monotonic sequence of impedances associated with the quarter-wave transformer. One is, therefore, led to the observation at this point that a symmetric coupler is less efficient in producing flat coupling than is the asymmetric variant of the quarter-wave transformer.

A pure transmission line cascade is always matched for $p=0$ and $L-1$, therefore, has at least a double root at the origin. Since the cascade, by intent, has the widest possible mismatch in frequency, good coupler design dictates that there be no other roots of $L-1$ on the imaginary axis for $|p|\leq 1$. In virtue of the results of the last section, the premises of coupler design yield the greatest number of basic root patterns and produce, in contrast to optimum quarter-wave transformers and transmission line filters, the greatest multiplicity of syntheses.

Multiple syntheses, as we shall show by example, do not as a rule maintain the monotonic impedance sequence ascribed to the quarter-wave transformer and thus are not as qualitatively evident from physical considerations in their application to couplers. One may gain some insight into the mechanism of these other solutions by the following argument. A transformer transforms a matched load into one having an essentially fixed value of reflection. An equally good solution for the coupler, however, is that transformation which takes a matched load into one having a substantially circular reflection locus in the reflection plane about the origin as a center. The degree of multiplicity of the variant syntheses relates to the possible number of admissible loci.

We may now show that multiple basic patterns occur only for $n\geq 3$. In discussing basic patterns we shall recall that imaginary roots of $L-1$ are simple for $|p|\geq 1$ and that simple root locus transformations may be employed to carry them through the point at infinity to the real axis. In our considerations, therefore, we shall make little distinction between real roots and those imaginary roots for $|p|\geq 1$.

If $n=1$ then $L-1$ is of second degree and the requirement of double roots at the origin requires that

$$L-1 \sim p^2.$$

If $n=2$, two roots are again consigned to the origin and the two remaining roots, not enough for a quartet, fall on the real axis. For both $n=1$ and 2, $k(p)$ is defined to within inversion symmetry of its roots so that there is but one basic pattern in each case.

If $n=3$, $L-1$ is of sixth degree and two types of root formations exist. 1) The roots, less the two at the origin, form a complex quartet still yielding but one basic root pattern. 2) The remaining four roots split up on the real axis; two unequal positive roots and their corresponding negatives. This last case yields two basic patterns as we may now observe. Let $p=0$, $a$, $b$ be the three independent roots of $L-1$. We may then have the following

$$k(p)=\frac{p\left(p-\sqrt{1+p^2}\,\dfrac{a}{\sqrt{1+a^2}}\right)\left(p-\sqrt{1+p^2}\,\dfrac{b}{\sqrt{1+b^2}}\right)}{D(p)} \quad (40a)$$

$$k(p)=\frac{p\left(p-\sqrt{1+p^2}\,\dfrac{a}{\sqrt{1+a^2}}\right)\left(p+\sqrt{1+p^2}\,\dfrac{b}{\sqrt{1+b^2}}\right)}{D(p)}. \quad (40b)$$

The root pattern chosen to form $k(p)$ in (40a) is 0, $a$, $b$, whereas the root pattern in (40b) is 0, $a$, $-b$.

For $n=4$, $L-1$ is of eighth degree and the root formation is for one case a pair at the origin, a quartet of complex roots, and a pair on the real axis. A more complex case might put more roots on the real axis and eliminate the complex quartet. In any event, $n>3$ always leads to multiplicity with a possible nonuniqueness occurring for $n=3$.

As a final comment on coupler design, there is at present no assurance that variant syntheses all lead to admissible coupler designs. Since the symmetric mode impedances are always greater than the corresponding impedances of the antisymmetric mode, a contradiction would occur if synthesis required a symmetric impedance less than unity since the dual would require an antisymmetric impedance greater than unity. An accessory requirement on coupler synthesis, therefore, is that all impedances of a symmetric mode array be greater than unity.

## VII. Example of Coupler Design

The identification of couplers with quarter-wave transformers motivates a choice of variables $v = \cos\theta$ which is more sensitive to a 90° structure than is $\sin\theta$. Let it be required to synthesize the loss function such that it is equal to a constant plus some even polynomial $f_{2n}(v)$. Since $L-1=0$ when $v=1$ the loss function takes the form

$$L = K - (K-1)\frac{f_{2n}(v)}{f_{2n}(1)} \tag{41}$$

where $f_{2n}(v) \leq f_{2n}(1)$ for $0 \leq v \leq 1$. For a Chebyshev departure $f_{2n}(v) = T_n^2(av)$, where $a \geq 1$, so that

$$L(v) = K - (K-1)\frac{T_n^2(av)}{T_n^2(a)}. \tag{42}$$

Let it be required to construct a four section Chebyshev coupler having a coupling of $3.01 \pm 0.1$ dB. We observe from (42) that

$$L_{\max} = K \rightarrow 3.11 \text{ dB}$$

$$L_{\min} = K - \frac{(K-1)}{T_n^2(a)} \rightarrow 2.91 \text{ dB}.$$

These two conditions provide

$$K = 2.046$$

and

$$T_4(a) = 8a^4 - 8a^2 + 1 = 3.37188.$$

We obtain[2] $a = 1.1132$. Since $av = \pm 1$ are the band edges of (42), we obtain these edges by the relationship $\cos\theta = \pm 1/a$. The coupler has a band ranging from $\theta = 26.06°$ to $\theta = 153.94°$; a spread of 5.9:1.

Since $p^2 = v^2 - 1$, we find from (42)

$$|k(p)|^2 = \frac{L-1}{L} = \frac{-p^2[13.885p^6 + 33.129p^4 + 27.383p^2 + 9.0916]}{1 - [13.885p^8 + 33.129p^6 + 27.383p^4 + 9.0916p^2]}. \tag{43}$$

The numerator roots are

$$p_i^2 = 0, \quad -1.19304, \quad \text{and} \quad -0.59648 \pm i.439366$$

while those of the denominator are

$$p_j^2 = 0.085576, \quad -1.278576, \quad \text{and} \quad -0.597 \pm i.549378.$$

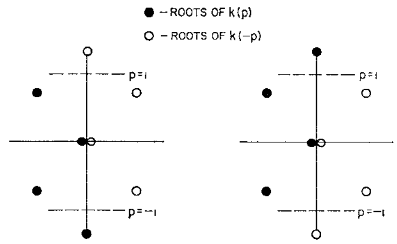



Fig. 5. Two basic patterns of $L-1$ for four section Chebyshev coupler.

The root patterns of $L-1$ are shown in Fig. 5 and we observe the two basic patterns anticipated in Section V (see Part I). The basic pattern on the left produces the matrix

$$T_4 = \begin{bmatrix} 16.206p^4 + 14.833p^2 + 1 & p\sqrt{1+p^2}(17.946p^2 + 8.41) \\ p\sqrt{1+p^2}(4.153p^2 + 2.38) & 4.599p^4 + 5.18p^2 + 1 \end{bmatrix} \tag{44}$$

corresponding to an impedance array

$$3.903, \; 2.03, \; 1.412, \quad \text{and} \quad 1.107 \tag{45}$$

while the pattern on the right produces the matrix

$$T_4' = \begin{bmatrix} 11.827p^4 + 12.4075p^2 + 1 & p\sqrt{1+p^2}(15.0388p^2 + 8.410) \\ p\sqrt{1+p^2}(7.0598p^2 + 2.3796) & 8.9782p^4 + 7.6055p^2 + 1 \end{bmatrix} \tag{46}$$

which leads to the array

$$1.6752, \; 4.1424, \; 1.3242, \; 1.2716. \tag{47}$$

[2] The authors apologize for their archaic use of slide rule and desk computer and advise the reader to take the numerical results with a small, but finite, grain of salt.

It is of interest to compare the differential phase between the coupled and transmission ports for both realizations. From (5) and (33) we find that

$$\Delta\phi = \arg \frac{k(p)}{t(p)} = \arg \left(M_n(p) + \sqrt{1+p^2}\, N_{n-1}(p)\right)$$

and that further reference to (11) and (12) permits reexpression of (48) as

$$\Delta\phi = \arg \left[(A - D) + (B - C)\right] \tag{49}$$

where $A$ and $D$, as employed earlier, are the respective upper and lower major diagonal terms of the matrix and $B$ and $C$ are the corresponding minor diagonal terms. From (44) we find

$$\Delta\phi = \arctan \left[ - \frac{(6.03 - 13.793 \sin^2\theta)\cot\theta}{9.653 - 11.607 \sin^2\theta} \right]. \tag{50}$$

The poles of the arctangent function occur for $\theta = 0°$ and $69°$ while the roots occur at $\theta = 41.4°$ and $90°$, with a symmetry of poles and roots about $\theta = 90°$. There is also an odd symmetry of $\Delta\phi$ about that value occurring for $\theta = 90°$.

Because of the alternation of poles and zeros of the arctangent it is an ever increasing function as shown in Fig. 6. Since (44) represents a quarter-wave transformer-like embodiment of the coupler, one roughly expects its approximation as a line length and terminating ideal transformer to lead to a uniformly increasing phase shift as a function of section length $\theta$, with a slope at band center corresponding to the order of the aggregate coupler length. Indeed, (50) shows a slope corresponding to a line length

$$\frac{13.793 - 6.03}{11.607 - 9.653}\,\theta = 3.97\theta,$$

which is an excellent approximation to $4\theta$.

Equation (44) provides a monotonic impedance sequence and provides intuitively pleasing results. Equation (46), on the other hand, leads to a nonmonotonic sequence and a consequent failure of intuition. The differential phase between coupled and transmission ports is

$$\Delta\varphi = \arctan \left[ - \frac{(6.0304 - 7.979 \sin^2\theta)\cot\theta}{4.802 - 2.8488 \sin^2\theta} \right]. \tag{51}$$

We find but one pole of the arctangent function at $\theta = 0°$ and zeros at $\theta = 60.4°$ and $90°$. Since there is a failure of alternation of poles and zeros, the slope of the differential phase function changes sign. The differential phase shift is shown in Fig. 7 with this observed property.

As a final point in this example of design we wish to show the invariance of $t(p)$ numerically for the two basic

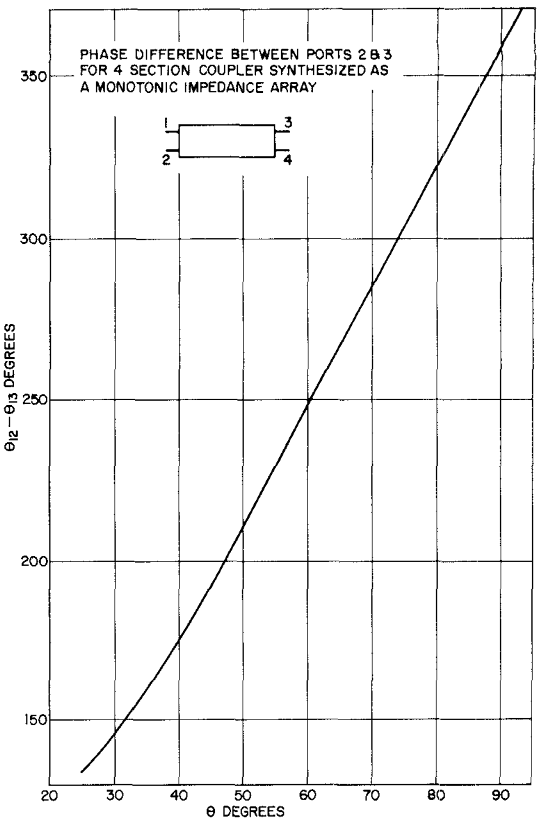



Fig. 6. Phase difference between ports 2 and 3 for four section coupler synthesized as a monotonic impedance array.

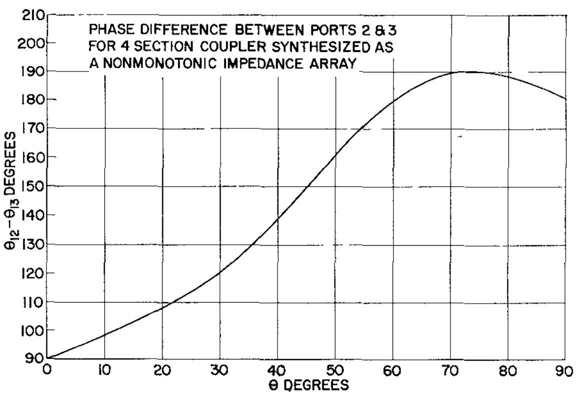



Fig. 7. Phase difference between ports 2 and 3 for four section coupler synthesized as a monotonic impedance array.

root patterns. Equation (33), in view of (11) and (12), meets the well-known relationship

$$t(p) = \frac{2}{A + B + C + D}\,. \tag{52}$$

Since the major and minor diagonal terms differ by an imaginary, the invariance of $t(p)$ implies that

$$\begin{aligned} A + D &= \text{Invariant} \\ B + C &= \text{Invariant}. \end{aligned} \tag{53}$$

Inspection of (44) and (46) reveal the correctness of (53).

## VIII. Symmetric Structures

A symmetric structure necessarily contains an odd number of sections and it is characterized by a major diagonal equality in (12). We arrive, therefore, at the necessary and sufficient condition for symmetry that

$$N_{n-1}(p) = 0. \tag{54}$$

Equation (8) yields the loss formulation

$$L(p) - 1 = - M_n^2(p). \tag{55}$$

It is instructive to show the form of (55) as following directly from (54). For $N_{n-1}(p)$ to vanish identically is to assert that there are no $\sqrt{1+p^2}$ multiplier terms in the numerator of $k(p)$ formed from the product $\prod_i (p-\sqrt{1+p^2}\alpha_i)$. This can only come about when for every $\alpha_i$ there exists a $-\alpha_i$ in the product. Hence, for every root of $k(p)$ its negative is contained as well so that $k(p)$ and $k(-p)$ have identical roots. Since all of the roots of $L-1$ are contained in the roots of $k(p)k(-p)$, the roots of $L(p)-1$ are double and (55) follows.

The sufficient condition to realize a symmetric structure, from the above arguments, is that which provides $k(p)$ and $k(-p)$ with identical roots. It is evident that this occurs when the numerator of $k(p)$ is $M_n(p)$ corresponding to a numerator of $k(-p)$ of $-M_n(p)$. This situation is shown graphically as well in Fig. 8(a). When the root equality of $k(p)$ and $k(-p)$ is modified, as in Fig. 8(b), an asymmetric structure results. Recapitulating the necessary and sufficient conditions for symmetric structures, we have for $n$ odd only and $M_n(p)$ an odd polynomial

1) $L=1-M_n^2(p)$
2) The roots of $k(p)$ and $k(-p)$ are identical.

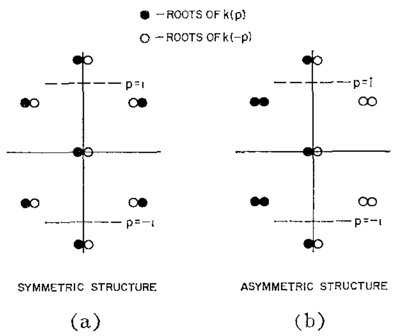



Fig. 8. Basic root pattern of $L(p)-1$ leading to a) symmetric, b) asymmetric realizations for loss function $L=1-M_n^2(p)$.

Given an odd polynomial of $n$th degree, $M_n(p)$, we have at our disposal $(n+1)/2$ constants that we may specify in $M_n^2(p)$. A more general polynomial of $2n$th degree permits a specification of $n$ constants so that the imposition of the constraint of symmetry reduces the degree of specification of band behavior to roughly half for $n$ adequately large. This result corresponds to the assertion in Section VI of the lower flatness "efficiency" of symmetric couplers in contrast to the optimally designed asymmetric structures.

We shall now show two means of synthesizing a symmetric coupler through the use of an example of the design of a three section coupler. The first method is admittedly approximate, but it has the virtue of producing a fairly acceptable result with a reasonable amount of calculation. The second method is exact and exacts more effort as well. As an extended example, an approximate five section coupler design analysis is also included.

### *A. Approximate Design of Three Section Symmetric Coupler*

A three section symmetric structure has two free constants and we choose a second order Chebyshev polynomial as the approximation function since it too has two constants. We then strive to satisfy the loss function

$$L = K - (K-1)\frac{T_2^2(av)}{T_2^2(a)} \tag{56}$$

where we recall the definition of $v$ as $v=\cos\theta=\sqrt{1+p^2}$. The loss function must simultaneously meet (55) and (56) so that we have

$$K - (K-1)\frac{T_2^2(av)}{T_2^2(a)} \simeq 1 - M_3^2(i\sqrt{1-v^2}). \tag{57}$$

Let $M_3(x)=C_1x+C_3x^3$, then

$$(K-1)\left[1-\frac{T_2^2(av)}{T_2^2(a)}\right] \simeq (1-v^2)(C_1-C_3(1-v^2))^2. \tag{58}$$

Equation (58) cannot be a true equality since the left side is a fourth degree in $v$ while the right side is of sixth degree. We choose the approximation such that as many lower degree terms as possible are equated. The loss function then differs from that specified by a high degree in $\cos\theta$ which is very small in the region about band center. This procedure yields

$$C_1 = \frac{3(3a^2-2)}{2a^2-1}\left(\frac{K-1}{a^2-1}\right)^{1/2} \tag{59a}$$

$$C_3 = \frac{a^3}{2a^2-1}\left(\frac{K-1}{a^2-1}\right)^{1/2}. \tag{59b}$$

Let us consider a 3-dB coupler having a 0.1-dB ripple. We obtain

$$K = 2.046, \qquad a = 1.4785,$$

$$C_1 = 1.8769, \quad \text{and} \quad C_2 = 0.90010$$

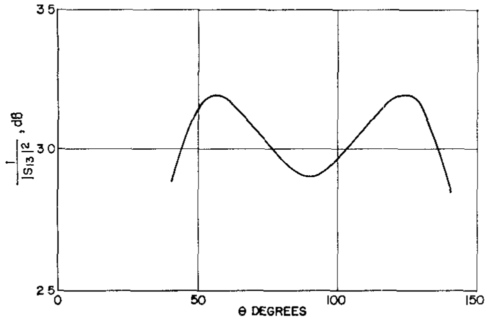
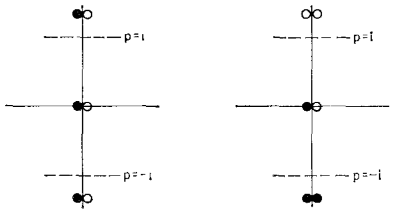



Fig. 9. Approximate three section symmetric coupler design.




(a) (b)

Fig. 10. Root patterns of three section coupler.

which lead to a loss function

$$L = 1 - p^2(0.90010p^2 + 1.8769)^2. \tag{60}$$

Figure 9 shows the results of (60) to adhere respectably to the desired result, possessing a ripple of $-0.1$ dB and $+0.2$ dB about 3 dB with a range of $\theta$ from 41° to 139°. The roots of $L-1$ form into the two patterns of Fig. 10 so that there exists a second, asymmetric, realization.

The symmetric structure has the root pattern of Fig. 10(a) and possesses the matrix representation

$$T_3 = \begin{bmatrix} \sqrt{1+p^2}(5.17925p^2+1) & 6.15681p^3 + 5.73464p \\ 4.35661p^3 + 1.98084p & \sqrt{1+p^2}(5.17925p^2+1) \end{bmatrix} \tag{61}$$

which provides the asymmetric impedance array

$$1.19, \quad 3.35, \quad 1.19. \tag{62}$$

The asymmetric realization of the loss function of (60) is obtained through the root pattern of Fig. 10(b). The numerator of $k(p)$ is now found to be

$$0.9768p[p + 1.38618\sqrt{1+p^2}]^2$$

so that the radical term $\sqrt{1+p^2}$ is introduced, confirming the asymmetric nature of the new structure. The new matrix representation is found to be

$$T_3' = \begin{bmatrix} \sqrt{1+p^2}(7.88729p^2+1) & 8.11041p^3 + 5.73464p \\ 2.40301p^3 + 1.98084p & \sqrt{1+p^2}(2.47121p^2+1) \end{bmatrix} \tag{63}$$

which provides the symmetric impedance array

$$3.28, \quad 1.421, \quad 1.028. \tag{64}$$

It is of interest to compare the differential phase between coupled and transmission ports for the two structures. Since the symmetric structure is characterized by the equality of the major diagonal terms in (61), (49) shows the differential phase to be equal to the argument of the difference of minor diagonal terms which leads to the well-known quadrature phase characteristics of the symmetric coupler. In particular, for the symmetric coupler,

$$\Delta\phi = \arg i(3.75380 - 1.80020 \sin^2 \theta). \tag{65}$$

Since there are no real roots of $\theta$ in (65), there is a constant lead by 90° of the coupled port with respect to the transmission port.

For the asymmetric structure of (63), (49) provides the differential phase

$$\Delta\phi = \arctan \frac{2.105 \sin^2 \theta - 1.383}{\sin 2\theta}. \tag{66}$$

The argument of the arctangent function has poles at 0° and 90° and a zero at 54°. Since the poles and zeros alternate, the arctangent function always has a positive slope. Here again, in correspondence with the results of the last section in relation to four section synthesis, the monotonic, or quarter-wave transformer like, realization of the loss function leads to a uniformly increasing differential phase with respect to $\theta$. The differential phase of the nonmonotonic impedance array does not correspond in any sense to an approximate line length. In particular the differential phase of the symmetric structure is absolutely flat.

The approximation method of design is extremely simple numerically and relaxation procedures readily converge on an exact solution. The constants $C_1$ and $C_3$ were obtained in a triangular fashion where the lowest order comparison with the approximate polynomial defined $C_1$, while the next order involved $C_1$ and $C_3$. Had there been still more constants in a synthesis of greater complexity, a progressively increasing involvement of constants would have occurred so that at each step only one constant is to be determined.

*B. Exact Design of Three Section Symmetric Coupler*

The exact design of the three section coupler is much more tedious arithmetically than is a relaxation method applied to the approximate design. It is included, however, for completeness.

Let $M_3(p)=A_3p^3+A_1p$. Its stationary points as a function of $\theta$ are given for

$$p = i, \qquad i\left(\frac{A_1}{3A_3}\right)^{1/2}.$$

The assumption of the existence of an oscillatory loss function implies that $1\geq(A_1/3A_3)\geq 0$. The loss at $p=i$ is given as

$$L(i) = 1 + (A_1 - A_3)^2 \tag{67}$$

$$L\left(i\left(\frac{A_1}{3A_3}\right)^{1/2}\right) = 1 + \frac{4}{27}\left(\frac{A_1^3}{A_3}\right). \tag{68}$$

We should like to have the loss contained between the stationary points at (67) and (68). We must be careful, however, that $M_3(p)$ has no root between these two points since this would introduce an undesired minimum in $L$ at a value of unity with a violent excursion of $L$ between the values 1 and $K$.

There exists the possibility that (67) is a minimum and (68) a maximum, and vice versa. We find, however, that a maximum at (67) brings with it a root in $M_3(p)$ for a real value of $\theta$, excluding that solution. There proves to be but one acceptable solution to the 3.01-dB $\pm$0.1-dB loss specification given by

$$A_1 = 1.79359; \qquad A_3 = 0.81539$$

so that the loss function is

$$L = 1 - (0.81539p^3 + 1.79359p)^2. \tag{69}$$

The symmetric matrix to (69) is

$$T_3 = \begin{bmatrix} \sqrt{1+p^2}(5.14286p^2+1) & 6.02250p^3 + 5.60182p \\ 4.39172p^3 + 2.01464p & \sqrt{1+p^2}(5.14286p^2+1) \end{bmatrix}. \tag{70}$$

with the corresponding impedance array

$$1.17104, \qquad 3.25979, \quad 1.17104. \tag{71}$$

Comparison of (61) and (62) with (70) and (71) shows the reasonableness of the zeroth order approximation employed.

The asymmetric realization has the matrix representation

$$T_3' = \begin{bmatrix} \sqrt{1+p^2}(7.79199p^2+1) & 7.97890p^3 + 5.60182p \\ 2.43533p^3 + 2.01465p & \sqrt{1+p^2}(2.49372p^2+1) \end{bmatrix}. \tag{72}$$

Corresponding to the impedance array

$$3.19958, \quad 1.37828, \quad 1.02398. \tag{73}$$

Again, the correspondence between (63) and (64) with (72) and (73) shows the approximation to be quite reasonable.

Figure 11 is a plot of the exact loss function and shows a band ranging from $\theta=45°$ to $\theta=135°$, for a frequency ratio of 3:1.

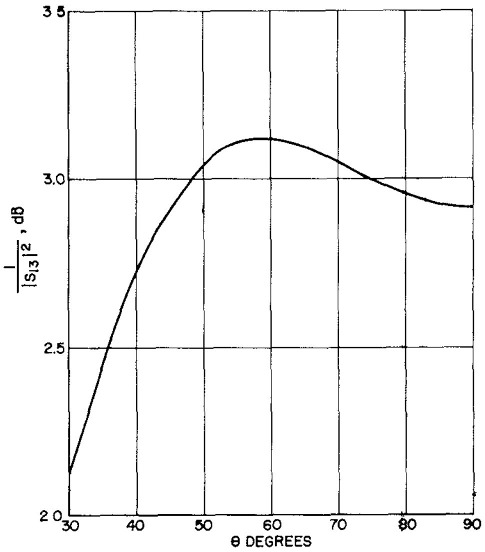



Fig. 11. Exact three section symmetric coupler design.

It is to be understood that this method is unlimited with respect to the polynomial degree of $L$ and that the choice of $n=3$ was taken only to provide an instructive example of the "exact" method.

*C. Approximate Design of Five Section Symmetric Coupler*

As an additional demonstration of the approximation method we seek to design a five section symmetric coupler. An approximation method which has been successful associates the stationary values of the loss function with the available degrees of freedom and an approximation function is sought with the appropriate number of stationary points. Since there are three undetermined coupler impedances and since the insertion loss function has mirror image symmetry about center band, there are five stationary points, implying a fourth order Chebyshev polynomial. In general, a Chebyshev polynomial of order $2n$ might be employed to approximate the response of a $2n+1$ section symmetric coupler.

The choice of the Chebyshev approximation and the matching of stationary values is but an election of the authors. There appears nothing optimal about this choice and other approximations with other fits and other polynomials are certainly valid.

The loss of a symmetric five section coupler is

$$L = 1 + M_5^2(u)$$

where, now, $u = \sin\theta$. We define $M_5(u)$ as

$$M_5(u) = C_5u^5 - C_3u^3 + C_1u$$

and seek the best fit to a loss function of the form

$$L \simeq 1 + (K-1)\left[1 - \frac{T_4^2(av)}{T_4^2(a)}\right]$$

where $v$, as before, is $\cos\theta$. If the coupler is assumed to have a 0.1 dB variation about 3.01 dB the lowest degree terms are determined by a matching procedure and we obtain

$$C_1 = 3.0164$$

$$C_3 = 4.5426$$

$$C_5 = 2.5491.$$

These values are the first rough approximation and, it should be pointed out, $C_5$ was obtained by matching the insertion loss at $u=1$.

The rough values above had to be refined and a relaxation method was employed up to a second perturbation. Final values obtained were

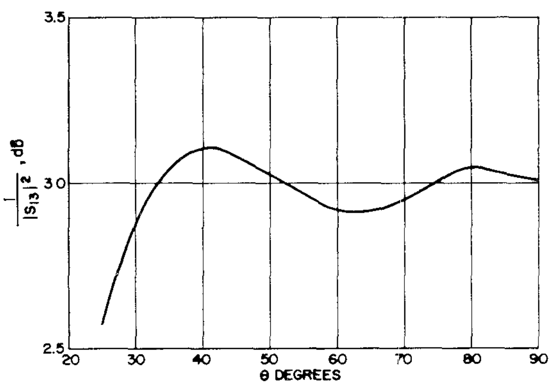



Fig. 12. Approximate five-section symmetric coupler design.

$$C_1 = 2.65779$$

$$C_3 = 3.24183$$

$$C_5 = 1.60700$$

with stationary values at

$$|u| = 0.890829, \qquad 0.645615.$$

The loss function is shown in Fig. 12 and displays good correlation to design specification.

## IX. Remarks on Transmission Line Structures

### *A. Equivalence of Filters and Quarter-Wave Transformers*

Filters and quarter-wave transformers have major similarities and it is the purpose of this section, employing the statement of realizability, to show a simple correlation.

An even loss function $L(p)$ is realizable if $L(p) \geq 1$ over the range $-1 \leq -ip \leq +1$. A transformation of $p^2$ to $-(1+p^2)$ neither modifies the evenness of $L(p)$ nor does it change the comain over which the inequality holds. Therefore, this transformation leads to an equally realizable structure. Let us designate $L(p)$ as the original loss function and $L'(p)$ as the loss function corresponding to the transformed variable. Then, $L(p)$ and $L'(p)$ are equal corresponding to a mathematically constant 90° difference of section length $\theta$. In particular, if $L = 1 + R_n^2(\sin\theta)$, then $L' = 1 + R_n^2(\cos\theta)$ and we arrive at the two types of loss functions given in (32a, b) which correspond to filters and quarter-wave transformers, respectively.

We characterize a filter as a pure transmission line array having no ideal terminating transformer, so that $L(0)=1$. Conversely, a quarter-wave transformer does have a terminating ideal transformer, that which is absorbed into the load mismatch, but has a match for $\theta = 90°$ so that $L'(i) = 1$. Let $L(p)$ be a filter loss function corresponding to an impedance array $Z_1, Z_2, \cdots, Z_n$. From Fig. 10 in [2] we find that the identical transmission and reflecting characteristics are obtained from an array

$$Z_1, \quad \frac{Z_1^2}{Z_2}, \quad \frac{Z_1^2}{Z_2^2}Z_3, \quad \frac{Z_1^2Z_3^2}{Z_2^2Z_4}, \cdots,$$

terminated by the transformer

$$\frac{Z_1Z_3Z_5\cdots}{Z_2Z_4Z_6\cdots} : 1,$$

where each section length $\theta$ is reduced to $\theta - 90°$, and where there is a 90° terminating section for $n$ odd. Since the tangent of the roots and poles $\theta_j$ of the reflection function of the filter correspond identically to their co-

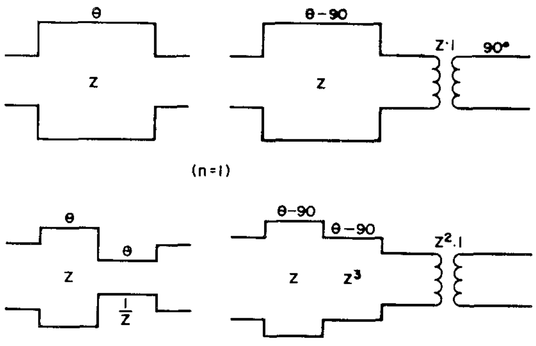



Fig. 13. Quarter-wave transformer and filter equivalents.

tangent in the new variable $\theta-90°$ for the identical reflection function in $\theta$ of the quarter-wave transformer, the quarter-wave transformer developed as indicated above from the filter prototype is exactly that which would be synthesized from the transformation $p^2 \rightarrow -(1+p^2)$.

Figure 13 shows examples of a quarter-wave transformer derived from a filter prototype for the cases of $n=1$ and 2. The results in Fig. 13 are quite well known. If in the case of $n=1$ we set $Z^2=R$ then we find for the one section transformer that an impedance $R$ is matched by a quarter-wave section of impedance $R^{\frac{1}{2}}$. Similarly in a two section case, a maximally flat match into an impedance $R$ is obtained by the cascade of sections having characteristic impedances, respectively, of $R^{\frac{1}{4}}$ and $R^{\frac{3}{4}}$.

### B. Application of Asymmetric Couplers to Mixers

Because of the lack of a simple phase relationship between $t(p)$ and $k(p)$ in an asymmetric coupler, one might question the utility of an asymmetric coupler in its application to a mixer where one hopes for a simple phase relationship between the intermediate frequency outputs at ports 2 and 3, respectively, in Fig. 14. We shall now show that there is a 180° intermediate phase difference between ports 2 and 3 when the IF frequency is relatively small.

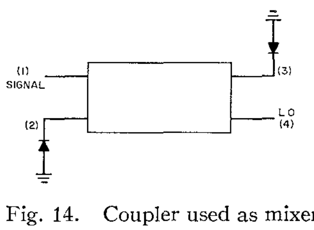



Fig. 14. Coupler used as mixer.

If the signal is applied to port 1 in Fig. 14 and the local oscillator to port 4, then if $\phi_2$ is the IF phase at port 2 and $\phi_3$ the IF phase at port 3, we have

$$\phi_2 = \arg \frac{S_{12}}{S_{42}} \tag{74a}$$

$$\phi_3 = \arg \frac{S_{13}}{S_{43}} \cdot \tag{74b}$$

We have assumed in (74) that the scattering coefficients are insensitive to the small IF frequency difference separating the signal and local oscillator. Identifying the scattering terms, we have $S_{12}=k(p)$, $S_{13}=S_{42}=t(p)$. It remains to identify $S_{43}$ in terms of $k(p)$ and $t(p)$.

The quantity $S_{43}$ is the even mode reflection coefficient of the reversed network. The even mode has the following two port scattering characterization,

$$\begin{pmatrix} S_{aa} & S_{ab} \\ S_{ab} & S_{bb} \end{pmatrix} = \begin{pmatrix} k(p) & t(p) \\ t(p) & S_{43} \end{pmatrix}. \tag{75}$$

The unitary property of the reactive scattering matrix provides

$$S_{43} = -\frac{t(p)}{t^*(p)} k^*(p). \tag{76}$$

Equations (74) and (76) combine to give

$$\Delta\phi = \phi_3 - \phi_2 = \arg \frac{S_{12}S_{43}}{S_{42}S_{13}} = \arg - \left(\frac{kk^*}{tt^*}\right) = 180°.$$

## X. Comments

Insertion loss synthesis through a cascade of equal length transmission lines goes back several years. From a purely human motivation it is the admittedly belated desire of one of the authors to set the record straight with respect to two of the earlier publications [1], [2]. H. J. Riblet [1] in his Discussion section makes two points about the doctoral dissertation of H. Seidel (Polytechnic Institute of Brooklyn, N. Y., May 1954):

1) "He (Seidel) does not introduce a complex variable equal to $p$ (equal to $-i \cot \theta$ in Riblet's notation) and thus does not have Richards' [7] theorem available for proving physical realizability."
2) "... he makes no point of the second condition for the physical realizability of an impedance function."

Working backwards in the order of objections, Riblet's "second condition" is that stated in (28) of the present paper, and it is seen to derive directly from the unity determinant condition of the transfer operator given in (13). The doctoral dissertation of 1954 is reproduced in essence in [2], and the unity determinant condition is given in (19) of that paper. The unity determinant condition followed as a consequence of the mode of realization which required only an assertion that the insertion loss function be of the form $L=1+R_n^2(\cos\theta)$. The very fact that the synthesis achieved was composed of reciprocal elements required this consequence.

As we have shown in the present paper, a more general condition of realizability is only the requirement that the loss function be even in $\cos\theta$ and always greater than or equal to unity. There is no requirement whatsoever on the sufficiency of this statement that there be a unity determinant, although a network reciprocity statement tantamount to a unit determinant is required to show the necessity and completeness of the synthesis. Riblet, in insisting that a special point must be made of his second condition, is in error for a statement of realizability based on insertion loss.

In response to the first objection cited, the choice of a PRF theoretic method of proof of realizability over a root locus approach is strictly fielder's choice. If any justification need be given for a choice of $p$ of the form of $\sin\theta$ as opposed to $\cot\theta$, it is given in Section IV of Part I of this paper. Nevertheless, roots of $\cot\theta$ were involved in [2] [(13b) of that paper] and a simple, adequate, root locus proof of realizability was given involving only the nature of the loss function.[3]

Instead of the two statements required by Riblet using a PRF mode of description to produce realizability only a single statement is required using an insertion loss description. This lack of economy of statement is reflected in a more recent paper by Levy which motivates him to state an incorrect theorem. It reads:

> Any insertion loss function of the form
>
> $$L = 1 + [f_1(a\cos\theta + b\sin\theta]^2 + [f_2(a\cos\theta + b\sin\theta)]^2$$
>
> can be realized as a stepped impedance filter with real positive characteristic impedances if the function $L=1+f_1^2(\omega)+f_2^2(\omega)$ having all its poles at infinity, is realizable as a two-port ladder network consisting of simple lossless series reactances and shunt susceptances terminated by resistances.[4]

That this theorem is incorrect may be observed simply by placing $f_1(\omega)=f_2(\omega)=\omega$ so that $L=1+2\omega^2$. This is realizable by a relative series reactance or shunt susceptance of value $2\sqrt{2}$ inserted between terminations. By Levy's theorem, $L=1+2(a\cos\theta+b\sin\theta)^2$ is equally realizable for all values of $a$ and $b$ in terms of a stepped impedance filter. Since the insertion loss of a reactive two-port is unaffected by time reversal it is even in $\theta$. This last loss function cannot be even unless either $a$ or $b$ is zero.

This theorem is unnecessary if we recognize that a necessary condition of realizability is that $f_1^2(\omega)+f_2^2(\omega)$ be even in $\omega$ together with the condition that $a$ or $b$ vanish. It is sufficient to meet the PRF conditions since $L$ is even in $\theta$ and greater than unity and it automatically meets Riblet's "second condition" since the prototype ladder structure has a transfer matrix determinant of unity. Little of substance is added to Riblet's two criteria for realizability. In the development of his couplers Levy chose the loss function

$$L = 1 + \beta^2 - h^2T_n^2(\omega)$$

where the transformation $\omega\to(\cos\theta/\cos\theta_0)$ fortunately produced no fundamental violations.

The use of a PRF description requires yet another restriction over the two of Riblet in that Young [8] adds a third to the effect that numerator and denominator polynomials of the impedance function be of the same degree. The completeness of realizability of an appropriate loss function guarantees this is so. The specific demonstration of this condition follows from (18) which shows that the first impedance of the stepped array is defined by a ratio of the leading coefficients of numerator and denominator, respectively, in the impedance function. A difference of degree would cause one of these coefficients to vanish, producing either a zero or an infinite characteristic impedance section. Since this necessarily implies infinite insertion loss, contrary to hypothesis, the polynomials are of equal degree.

It is not without reason that a positive real function theory approach requires a greater multiplicity of restrictions for realizability than does the insertion loss statement using a root locus procedure. PRF theory is very general and covers many classes of structures of which the cascaded transmission line structure is but one. It is, therefore, necessary to impose these added restrictions to diminish the initial excessive generality. The insertion loss function, on the other hand, contains within its very formulation the restrictions associated with this class of structures and, as we have shown, is adequate for realizability to within the two obvious physical restrictions that it 1) be passive ($L\geq 1$), 2) be time reversible ($L(i\sin\theta)=L(-i\sin\theta)$).

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